Cross Product of two Vectors

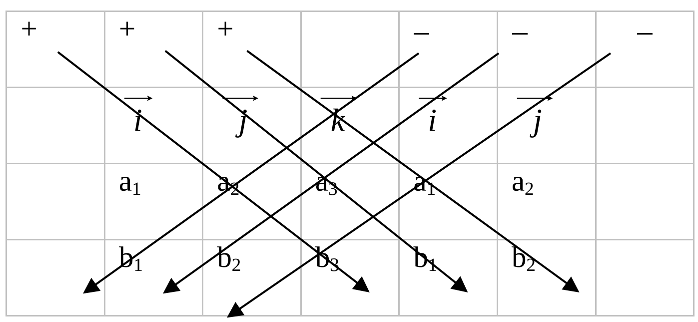
> **Def^{n}:** Let A = (a_1, a_2, a_3) and B = (b_1, b_2, b_3) be two vectors. The **cross (Vector) product** of A and B, written A × B is defined by:
>
> $$A \times B = (a_2 b_3 - b_2 a_3)\,\vec{i} + (a_3 b_1 - a_1 b_3)\,\vec{j} + (a_1 b_2 - a_2 b_1)\,\vec{k}$$
>
> A × B is read as "A cross B".

Now let us see a simple method how to recall the formula for the cross product of A and B

i) The first method.

$$A \times B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

ii) The second method.



Example: Let A = (5, – 1, 0) and B = (0, 2, – 2). Find A × B and B × A.

Solution:

$$A \times B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -1 & 0 \\ 0 & 2 & -2 \end{vmatrix}$$

$$= 2\,\vec{i} + 10\,\vec{j} + 10\,\vec{k}$$

Therefore $A \times B = 2\,\vec{i} + 10\,\vec{j} + 10\,\vec{k}$.

$$A \times B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & -2 \\ 5 & -1 & 0 \end{vmatrix}$$

$$= -2\,\vec{i} - 10\,\vec{j} - 10\,\vec{k}$$

Therefore $A \times B = -2\,\vec{i} - 10\,\vec{j} - 10\,\vec{k}$.

Remark: $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$ and $\vec{k} \times \vec{i} = \vec{j}$.

Properties of Cross Product

Let A, B, C be vectors and let m be a scalar. Then

i) $A \times B = -(B \times A)$

ii) $A \times A = \vec{0}$

iii) $A \times (B + C) = (A \times B) + (A \times C)$

and $(A + B) \times C) = (A \times C) + (B \times C)$

iv) $(m A) \times B = m (A \times B) = A \times (m B)$.

Remark: Cross Product is not associative.

Example: $\vec{i} \times (\vec{k} \times \vec{k}) = \vec{0}$ while, $(\vec{i} \times \vec{k}) \times \vec{k} = -\vec{j} \times \vec{k} = -\vec{i}$.

Theorem: Let A and B be two non-zero vectors.

a) $A \cdot (A \times B) = 0$ and $B \cdot (A \times B) = 0$

Consequently; if $A \times B \neq \vec{0}$, then $A \times B$ is orthogonal to both A and B.

b) If θ is the angle between A and B ($0 \le \theta \le \pi$), then

$$|A \times B| = |A|\,|B| \sin\theta.$$

Proof: i) $A \cdot (A \times B) = a_1 (a_2 b_3 - b_2 a_3) + a_2 (a_3 b_1 - a_1 b_3) + a_3 (a_1 b_2 - a_2 b_1)$

$= a_1a_2\, b_3 - a_1a_3b_2 + \; a_2a_3\, b_1 - a_1a_2b_3 + a_1a_3b_2 - a_2\, a_3b_1$

$= (a_1a_2\, b_3 - a_1a_2b_3) + (a_1a_3b_2 - a_1a_3b_2) + (a_2a_3\, b_1 + - a_2\, a_3b_1)$

$= 0.$

$B \cdot (A \times B) = b_1\,(a_2\, b_3 - b_2a_3) + \; b_2\,(a_3\, b_1 - a_1\, b_3) + b_3\,(a_1\, b_2 - a_2\, b_1)$

$= a_2\, b_1b_3 - a_3\, b_1b_2 + \; a_3b_1b_2 - a_1b_2b_3 + a_1b_2b_3 - a_2b_1b_3$

$= (a_2\, b_1b_3 - a_2b_1b_3) + (a_3b_1b_2 - a_3\, b_1b_2) + \; (a_1b_2b_3 - a_1b_2b_3)$

$= 0.$

ii) $|A \times B|^2 = (a_2\, b_3 - b_2a_3)^2 + (a_3\, b_1 - a_1\, b_3)^2 + (a_1\, b_2 - a_2\, b_1)^2$

$= (a_2^2 b_3^2 - 2\, a_2a_3b_2b_3 + b_2^2\, a_3^2) + (a_1^2\, b_3^2 - 2\, a_1a_3b_1b_3 + b_1^2\, a_3^2)$

$+ (a_1^2 b_2^2 - 2\, a_1a_2b_1b_2 + a_2^2 b_1^2)$

$= a_1^2\,(b_2^2 + b_3^2) + a_2^2\,(b_1^2 + b_3^2) + a_3^2\,(b_1^2 + b_2^2)$

$- \; (2\, a_2a_3b_2b_3 + 2\, a_1a_3b_1b_3 + 2\, a_1a_2b_1b_2)$

$= (a_1^2 + a_2^2 + a_3^2)\; (b_1^2 + b_2^2 + b_3^2) - \; (a_1b_1 + a_2b_2 + a_3b_3)^2$

$= \; |A|^2 |B|^2 - (|A|\,|B| \cos\theta)^2$

$= |A|^2 |B|^2 - \; |A|^2 |B|^2 \cos^2\theta$

$= |A|^2 |B|^2 (1 - \cos^2\theta)$

$= |A|^2 |B|^2 \sin^2\theta$

Therefore $|A \times B| = |A|\,|B| \sin\theta.$

Corollary: Two non-zero vectors A and B are parallel if and only if $A \times B = \overrightarrow{0}$.

Proof: $A \times B = \overrightarrow{0} \Leftrightarrow |A \times B| = 0$

$\Leftrightarrow |A|\,|B| \sin\theta$

$\Leftrightarrow \sin\theta = 0$

$\Leftrightarrow \theta = 0$ or $\theta = \pi$.

$\Leftrightarrow A \parallel B.$

Therefore Two non-zero vectors A and B are parallel if and only if $A \times B = \overrightarrow{0}$.

Example: Let A (3, 2, – 2) and B (0, 3, 7).

i) Determine whether A and B are parallel or orthogonal or neither.

ii) Find a vector orthogonal to both A and B.

Solution: i) $A \cdot B = (3 \times 0) + (2 \times 3) + (-2 \times 7) = -2$, $|A| = \sqrt{17}$ and $|B| = 2\sqrt{14}$.

Hence neither $A \cdot B \neq 0$ nor $|A \cdot B| \neq |A||B|$.

Therefore A and B are neither parallel nor orthogonal.

Remark: $|A \times B|$ is the area of a parallelogram with adjacent sides A and B.

Triple Product

There are two types of triple products.

i) **Scalar triple product**

For any three vectors A, B and C, $A \cdot (B \times C)$ is called the **triple** (**box** or **mixed triple**) product of A, B and C.

Example: Show that for any three vectors A, B and C

$$A \cdot (B \times C) = (A \times B) \cdot C$$

Solution: $A \cdot (B \times C) = |A||B \times C| \cos\alpha$, where α is the angle between A and $B \times C$.

$= |A||B||C| \cos\alpha \sin\beta$, where β is the angle between B and C

and $(A \times B) \cdot C = |A||B||C| \cos\psi \sin\upsilon$, where ψ is the angle between C and $A \times B$ and υ is the angle between A and B.

Now $\cos\alpha = \sin\upsilon$ and $\sin\beta = \cos\psi$, because co-functions of complementary angles are equal.

Therefore $A \cdot (B \times C) = (A \times B) \cdot C$.

ii) **Vector Triple Product**

For any three vectors A, B and C, $A \times (B \times C)$ is called the **Vector triple** product of A, B and C.

Example: Show that for any three vectors A, B and C

$$A \times (B \times C) \neq (A \times B) \times C$$

Solution: $A \times (B \times C) = |A||B \times C| \sin\alpha$, where α is the angle between A and $B \times C$.

$= |A||B|\ |C|\ \sin\alpha \sin\beta$, where β is the angle between B and C

and $(A \times B) \times C = |A||B|\ |C|\ \sin\psi \sin\upsilon$, where ψ is the angle between C and $A \times B$ and υ is the angle between A and B.

Now $\sin\alpha \neq \sin\upsilon$ and $\sin\beta \neq \sin\psi$.

Therefore $A \times (B \times C) \neq (A \times B) \times C$.

Remark: For any three vectors A, B and C

$$A \times (B \cdot C), (A \cdot B) \times C \text{ and } (A \cdot B) \cdot C$$

are undefined operations.

Some Properties of Triple Products

For any three vectors A, B and C

i) $(A \times B) \cdot C = B \cdot (C \times A) = A \cdot (B \times C)$

ii) $A \times (B \times C) = B\,(A \cdot C) - C\,(A \cdot B)$ “ bac – cab” rule.

Remark: For any three non-zero vectors A, B and C; $|A \cdot (B \times C)|$ is the **volume** of a parallelepiped with sides A, B and C.